Orthogonal Arrays (OAs)

- **Definition:** An OA of strength \( t \) is a two-dimensional array, table, or matrix of \( N \) rows and \( k \) columns:
  - Each entry is one of a set of \( s \) “symbols”, often taken to be \( \{0, 1, 2, ..., s - 1\} \).
  - Every subset of \( t \) columns (from among the \( k \) columns), when considered alone, must contain each of the possible \( s^t \) ordered rows the same number of times.

- **Notation:** \( OA(N, k, s, t) \).

- **Example:** \( OA(9, 4, 3, 2) \):

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 1 & 1 & 2 \\
0 & 2 & 2 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 2 & 0 \\
1 & 2 & 0 & 2 \\
2 & 0 & 2 & 2 \\
2 & 1 & 0 & 1 \\
2 & 2 & 1 & 0 \\
\end{array}
\]
Connection to what you’ve seen:

- An OA of strength \( t = k \) is a full factorial design.
- Regular fractional factorials are a sub-class of OAs.
- A “projective property” of an OA of strength \( t \) is that it is a full factorial design in any \( t \) factors, ignoring the rest.

Index:

- The above is an example of an OA of index 1 (or \( \lambda = 1 \)) because each subset of \( t = 2 \) columns contains each of the possible \( s^t = 9 \) rows once.
- So, for example, repeating this array would result in an \( OA(18, 4, 3, 2) \) with \( \lambda = 2 \) (but not all OAs with \( \lambda > 1 \) are constructed this way).
- \( \lambda \) is closely associated with resolution in regular fractions.
Orthogonality (2 ways):

- “Column orthogonality”
  - Two columns of $s$ symbols are orthogonal if each of the $s^2$ (ordered) pairs of symbols appears in an equal number of rows.
  - Likewise, pairs of sets of columns can be orthogonal. For example, in OAs of strength 4, every pair of columns is orthogonal to every other pair; every pair of ordered pairs of symbols appears in an equal number of rows.

- “Contrast orthogonality”
  - Two data contrasts are orthogonal if the inner product of the weight vectors is zero.
• A connection (by example):
  − Consider a design in which each factor has 3 (say) levels, and for which the columns corresponding to factors 1 and 2 are, as a set, orthogonal to the columns corresponding to factors 3, 4, and 5.
  − Specify any set of 9 weights that sum to zero, and form a data contrast with these by defining a correspondence between the weights and the 9 paired values of factors 1 and 2.
  − Similarly, construct a set of 27 weights for a data contrast associated with the ordered values of factors 3, 4, and 5.
  − Then the two data contrasts just generated are orthogonal.

Statistical motivation for the use of OAs as designs comes from the fact that their (column) orthogonal structure implies the (contrast) orthogonal structure of factorial effect estimates.
**Statistical Motivation**

Unambiguous interpretation of the data from fractional factorials (including OAs) requires assumptions, most commonly that interactions above a specified order are not present.

- OAs of even strength $t = 2u$:
  - Every pair of sets of columns with $u$ or fewer columns is orthogonal.
  - Therefore, every pair of factorial effect contrasts through order $u$ is orthogonal. As a consequence, a model that contains all factorial effects through order $u$ can be fitted with maximum efficiency, and the estimates of these model terms are unbiased if all interactions of order $u + 1$ and higher are absent.
  - So, OAs of strength 2, 4, 6, ... correspond to regular fractions of resolution 3, 5, 7, ..., respectively.
• OAs of odd strength \( t = 2u + 1 \):
  
  - Every pair of sets of columns with \( u \) or fewer columns is orthogonal, and every set of \( u \) columns is orthogonal to every set of \( u + 1 \) columns.
  
  - Therefore, every pair of factorial effect contrasts through order \( u \) is orthogonal, and every factorial contrast of order \( u \) is orthogonal to every factorial contrast of order \( u + 1 \). As a consequence, a model that contains all factorial effects through order \( u \) can be fitted with maximum efficiency, and the estimates of these model terms are unbiased if all interactions of order \( u + 2 \) and higher are absent.
  
  - So, OAs of strength 3, 5, 7, ... correspond to regular fractions of resolution 4, 6, 8, ..., respectively.
But not all OAs are regular fractions, because the latter are defined under more constraints:

- *Every* pair of factorial effects (contrasts) of any order in a regular fraction, regardless of resolution, is required to be either orthogonal or (completely) confounded.

- *Only* pairs of OA column groups of size $u_1$ and $u_2$ with $u_1 + u_2 = t$ are required to be orthogonal, implying that only factorial effect contrasts through orders $u_1$ and $u_2$ need to be orthogonal.

- There is no structural requirement in OAs for the relationship between sets of $u_1$ and $u_2$ columns with $u_1 + u_2 > t$.

- E.g., for the 12-run Plackett-Burman design ($s = 2$), the main effect contrast associated with any factor is neither orthogonal to nor completely confounded with two-factor interaction contrasts associated with any other two factors.
Restrictions/Bounds on the size (N) of OAs

• For any OA(N, k, s, t), \( N = \lambda \times s^t \).

• Rao’s (1947) inequalities (serve either as an upper bound on the number of columns that can be included in an OA of given \( N, s, \) and \( t \); or a lower bound on the number of rows required for an OA of given \( k, s, \) and \( t \)):

\[
N \geq \sum_{i=0}^{u} \binom{k}{i} (s - 1)^i \quad \text{if } t = 2u
\]

\[
N \geq \sum_{i=0}^{u} \binom{k}{i} (s - 1)^i + \binom{k-1}{u} (s - 1)^{u+1} \quad \text{if } t = 2u + 1
\]

One proof uses the relationship between “orthogonal columns” and “orthogonal contrasts”, e.g. the RHS of the first inequality is the number of parameters in a first-order model when \( t = 2 \). (See notes for details, or HSS for even more.)
A Connection Between OAs and Latin Squares

Recall that a Latin Square of order $s$ is an $s \times s$ array of $s$ symbols, each appearing $s$ times, once in each row and once in each column. Label the $s$ rows and $s$ columns by the same set of “symbols” used as table entries, and form an array in which each row contains the row-block symbol, the column-block symbol, and the Latin Square treatment symbol for a cell in the table, this is an $OA(s^2, 3, s, 2)$, e.g.:

<table>
<thead>
<tr>
<th>row</th>
<th>col</th>
<th>“trt”</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
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<tr>
<td>1</td>
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<td>1</td>
<td>1</td>
<td>2</td>
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<tr>
<td>1</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

\[
\begin{array}{ccc}
0 & 1 & 2 \\
1 & 2 & 0 \\
2 & 0 & 1 \\
\end{array}
\quad \rightarrow 
\begin{array}{ccc}
0 & 0 & 0 \\
0 & 1 & 1 \\
0 & 2 & 2 \\
1 & 0 & 1 \\
1 & 1 & 2 \\
1 & 2 & 0 \\
2 & 0 & 2 \\
2 & 1 & 0 \\
2 & 2 & 1 \\
\end{array}
\]
Recall also that Graeco-Latin Square designs are constructed by using two *mutually orthogonal Latin Squares* of the same order, with the property that when overlaid, each symbol in the body of one square appears together with each symbol in the body of the other exactly once. The second square allows us to add a 4th column to the orthogonal array, yielding an $OA(s^2, 4, s, 2)$, e.g.:

\[
\begin{array}{ccc}
0 & 1 & 2 \\
1 & 2 & 0 \\
2 & 0 & 1 \\
\end{array}, \quad \begin{array}{ccc}
0 & 2 & 1 \\
1 & 0 & 2 \\
2 & 1 & 0 \\
\end{array} \quad \rightarrow \quad \begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 1 & 1 & 2 \\
0 & 2 & 2 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 2 & 0 \\
1 & 2 & 0 & 2 \\
2 & 0 & 2 & 2 \\
2 & 1 & 0 & 1 \\
2 & 2 & 1 & 0 \\
\end{array}
\]

Generally, this process can be extended to $s - 1$ pairwise orthogonal Latin Squares of order $s$ *when $s$ is a prime number*. Hence for prime $s$, single-index (i.e. $\lambda = 1$) orthogonal arrays of strength 2 can be constructed in up to $k = s + 1$ columns.
**Basics of Galois Fields**

Recall that a *field* is composed of

- A *set* $F$
- Two binary operations (which we will call “$+$” and “$\times$”) that map $F \times F$ into $F$, and satisfy the following for every $a, b, c \in F$:
  - *(Commutative)* $a + b = b + a$ and $a \times b = b \times a$.
  - *(Associative)* $(a + b) + c = a + (b + c)$ and $(a \times b) \times c = a \times (b \times c)$.
  - *(Identity)* There are unique elements 0 and 1 of $F$ such that, for any $a \in F$, $a + 0 = a$ and $a \times 1 = a$.
  - *(Inverse)* There is a unique $-a$ such that $a + (-a) = 0$, and there is a unique $a^{-1}$ such that $a \times a^{-1} = 1$.
  - *(Distributive)* $a \times (b + c) = (a \times b) + (a \times c)$.

- A *Galois* field is one for which the set $F$ is finite; $\#(F)$ is the *order* of the field.
Construction of any Galois field essentially amounts to construction of two square tables, the size of which is the field order, that define $+$ and $\times$. The simplest construction, and one that is very convenient for our purposes, is the collection of Galois fields for which:

- $F = \{0, 1, 2, \ldots, p - 1\}$, where $p$ must be a prime number
- $+$ will be addition, modulo $p$; e.g. for $p = 3$, $2 + 1 = 0$.
- $\times$ will be multiplication, modulo $p$; e.g. for $p = 3$, $2 \times 2 = 1$.

For this field, the elements we are calling 0 and 1 are, as in ordinary arithmetic, the additive and multiplicative identities, respectively. Hence, the entire Galois field of this type is determined by the value of $p$, and we refer to this specific field as $GF^*(p)$ to denote its special structure.
p not prime

• Galois fields do exist, however, for cases in which the order is not a prime number. (See notes for an example.)

• For Galois fields with order $o$ not prime, there is a constraint on the variety of structures that can be realized. One expression of this constraint is through the so-called integral elements of the field, which are defined using the multiplicative inverse element and the + operation, as the unique values of:

$$1, 1 + 1, 1 + 1 + 1, \ldots$$

• The number of integral elements is the characteristic of the field.

• For the $GF^*$ construction above, the order equals the characteristic (which must be prime). Two facts that are true for any Galois field are:
  – The characteristic $c$ is a prime number.
  – The order $o = c^n$ for some integer $n$. 
Polynomials over fields

- For a given Galois field, we denote by $F[x]$ the collection of all polynomials (in a variable we’re calling $x$) that can be formed with coefficients from the set $F$. For example, using $GF(5)$, one of the elements of $F[x]$ is:

$$3 + 2 \times x + 0 \times x^2 + 4 \times x^3.$$ 

- For notational convenience, we’ll also write this with the usual convention of omitting the symbol for multiplication:

$$3 + 2x + 0x^2 + 4x^3.$$ 

- In this context, $x$ denotes an arbitrary element of $F$, and evaluation of the polynomial proceeds by using the definition of $+$ and $\times$ specified for the field; for $GF^*(s)$, evaluation can proceed via the usual rules for addition and subtraction, and the result then reduced modulo $s$. 

OA Construction Methods Based on Galois Fields

Bush Construction 1

• For $s$ a prime power $\geq 2$, and $s \geq t - 1 \geq 0$, an $OA(s^t, s + 1, s, t)$.

• Construct an $s^t$-by-$s$ array (i.e. all but the last column) by giving a label to each column and row:
  - The $s$ columns are labeled with the $s$ elements of $GF(s)$.
  - The $s^t$ rows are labeled with the elements of $F[x]$ of degree up to $t - 1$ (i.e. for which each of the $t$ coefficients is one of the elements of $F$).

• The $(i, j)$ element of this array is then found by evaluating the polynomial that is the label for row $i$ at $x =$ the label for column $j$. Elements of the final column of the OA are defined to be the coefficients of $x^{t-1}$ in the corresponding (row-label) polynomials.
Example

Construct an OA in \( s = 3 \) symbols of strength \( t = 2 \), in \( N = s^t = 9 \) rows and \( k = s + 1 = 4 \) columns. Use \( GF^*(3) \):

<table>
<thead>
<tr>
<th>( F[x] )</th>
<th>( x = 0 )</th>
<th>( x = 1 )</th>
<th>( x = 2 )</th>
<th>coefficient of ( x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 + 0x</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0 + 1x</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>0 + 2x</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1 + 0x</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1 + 1x</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1 + 2x</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>2 + 0x</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>2 + 1x</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2 + 2x</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>
Bush Construction 2

- For $s = 2^m$ with $m \geq 1$, an $OA(s^3, s + 2, s, 3)$.
- Use Construction 1 to obtain an $OA(s^{3m}, 2^m + 1, 2^m, 3)$.
- Adjoin an additional column for which elements are defined to be the coefficients of $x$ (first power) in the corresponding (row-label) polynomials.

Hence, Construction 2 allows an extra column to be added to the array, relative to what Construction 1 provides, in the special case that $s$ is a power of 2.
Example

Construct an OA in $s = 4 = 2^2$ symbols of strength $t = 3$, in $N = 4^3 = 64$ rows and $k = 4 + 2 = 6$ columns.

- Begin by using Construction 1 to find an $OA(64, 5, 4, 3)$ using a $GF(4)$ – here’s one:

$$
\begin{array}{c|cccc}
+ & 0 & 1 & 2 & 3 \\
0 & 0 & 1 & 2 & 3 \\
1 & 1 & 0 & 3 & 2 \\
2 & 2 & 3 & 0 & 1 \\
3 & 3 & 2 & 1 & 0 \\
\end{array}
\quad
\begin{array}{c|cccc}
\times & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 2 & 3 \\
2 & 0 & 2 & 3 & 1 \\
3 & 0 & 3 & 1 & 2 \\
\end{array}
$$

- Augment with the last column as indicated.
<table>
<thead>
<tr>
<th></th>
<th>$x = 0$</th>
<th>$x = 1$</th>
<th>$x = 2$</th>
<th>$x = 3$</th>
<th>Coefficient of $x^2$</th>
<th>Coefficient of $x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 + 0x + 0x^2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$0 + 0x + 1x^2$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$0 + 0x + 2x^2$</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>$0 + 0x + 3x^2$</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>3</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>$0 + 1x + 0x^2$</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$3 + 3x + 3x^2$</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>
Rao-Hamming Construction

- For $s$ a prime power, an $OA(s^n, (s^n - 1)/(s - 1), s, 2)$, for any $n \geq 2$.

- Construct an $s^n \times n$ array $C$ comprised of rows that are all possible $n$-tuples of elements from $GF(s)$.

- Let $z$ be an $n$-element column vector with elements from $GF(s)$, and such that not all elements are 0’s, and the first nonzero is 1. (Recall that this is the same set of restrictions we called the “uniqueness condition” for sets of coefficients in the equations we used in generating blocking structures for $p^f$ experiments in STAT 512.)

- Then each column of the OA is generated as the product $Cz$; there are $(s^n - 1)/(s - 1)$ such vectors $z$, as required for the construction.
**Example**

Construct an OA in \( s = 2 \) symbols of strength \( t = 2 \), in \( N = 2^3 = 8 \) rows and \( (2^3 - 1)/(2 - 1) = 7 \) columns by starting with the array of all 8 possible 3-tuples from \( GF(2) \), and post-multiplying by the 7 vectors \( z \) satisfying the stated conditions:

\[
C = \begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
1 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 1 \\
0 & 1 & 1 \\
1 & 1 & 1 \\
\end{pmatrix}, \quad z = \begin{pmatrix}
0 \\
0 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
\end{pmatrix} \rightarrow \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 1 \\
\end{pmatrix}
\]
Mixed-Level OAs

OAs can be generalized by allowing different numbers of symbols in each column, while retaining the requirement that every subset of \( t \) columns contain each possible ordered \( t \)-element row the same number of times.

- A small example of an \( OA(8, 2^2, 4^1, 2) \):

<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- The first two columns contain each of the possible rows – \((0, 0), (0, 1), (1, 0), (1, 1)\) – twice, while either of the first two columns and the third contain each possible ordered pair of values once.
**Mixed OA Construction**

- A number of algebraic techniques have been developed for constructing mixed OAs (and many of these are discussed in HSS).

- Many of these are relatively specialized, e.g. are only relevant for a particular number of symbols for some factors, and most require that a simpler OA (perhaps with all columns containing the same number of symbols) be specified as a “seed array”.

- Computational algorithms are also useful in constructing mixed OAs in some cases. The following is a brief description of one such algorithm.
DeCock-Stufken Algorithm

DeCock and Stufken (2000) described an algorithm for constructing mixed strength-2 OAs in most columns contain 2 symbols, and one or a few contain more. (This situation is not uncommon in screening experiments, where many factors can reasonably be represented at two levels, but a few require 3 or more.) Here, we follow their description of constructing an OA$(N, 2^k, 3^1)$, where $N$ is specified (and must be a multiple of both 4 and 6!), and the object is to construct an OA with the largest possible value of $k$; the extension to $s_2 \neq 3$ should be fairly obvious.
The key to the DeCock-Stufken algorithm is the observation that the array to be constructed can be represented as:

\[
\begin{align*}
C_0 & \quad 0 \\
C_1 & \quad 1 \\
C_2 & \quad 2
\end{align*}
\]

where 0, 1, and 2 are \( \frac{N}{3} \)-element column vectors of 0’s, 1’s and 2’s, respectively; and \( C_0, C_1, \) and \( C_2 \) are \( \frac{N}{3} \times k \) arrays of 0’s and 1’s such that:

- Each \( C_i \) is an OA of strength 1, and
- The \( N \times k \) row-juxtaposition of \( C_0, C_1, \) and \( C_2 \) – call this array \( C \) – is an OA of strength 2.

The first requirement assures that each pair of columns including the last meets the strength-2 orthogonality requirement, and the second assures that all two-symbol columns meet the requirement.
• Specify a $OA(N, k^*, 2, 2)$ “seed array” $A^*$ for the required value of $N$, and with the maximum possible value of $k^*$. (e.g. Plackett-Burman designs are strength-2 2-level OAs with maximal number of columns for any value of $N$ that is a multiple of 4.)

• Randomly sample (without replacement) $N/3$ rows from $A^*$ and determining the number of columns in that sample that are balanced (i.e. contain the same number of 0’s and 1’s). Repeat a large number of times; the sample of rows for which the number of balanced columns is largest is retained. The unbalanced columns are removed, and the resulting $N/3 \times k_0$ array is saved ... call this $B_0$. (Note that $k_0 \leq k^*$, but the intent is that $k_0$ is a large as possible.)
Those rows of \( A^* \) that are represented in \( B_0 \) are removed, and the columns of \( A^* \) that were discarded in constructing \( B_0 \) are also removed. Refer to the resulting \( 2N/3 \times k_0 \) matrix as \( A^{**} \). Use the same sampling procedure described above, with the aim of finding the sample of \( N/3 \) rows of \( A^{**} \) with the largest number of balanced columns. Suppose the best sample found has \( k \) such columns (which clearly can be no larger than \( k_0 \)). Eliminate the unbalanced columns from this sample and from \( B_0 \) ... that is, retain only columns from the original seed array which are balanced in both samples, and call these two \( N/3 \times k \) arrays \( C_0 \) and \( C_1 \).

Finally, define the \( N/3 \times k \) array \( C_2 \) to be those rows from \( A^{**} \) that were not used to generate \( C_0 \) or \( C_1 \), after eliminating all columns that were excluded from the two sampling steps. It should be obvious that \( C_2 \) is also an OA of strength 1.
The product of the search is, then, an $OA(N, 2^k, 3^1, 2)$. As with most stochastic algorithms, there is no guarantee that the constructed OA actually has the largest possible number of 2-symbol columns for the problem specifications. But for large enough samples, the algorithm tends to work well in many situations.